Simons Collaboration on Computational Biogeochemical Modeling of Marine Ecosystems
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A Stochastic Lagrangian Perspective on Primary Production

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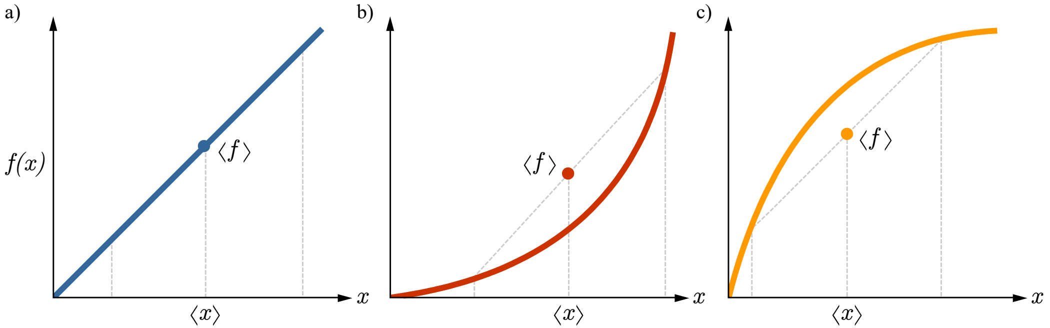
This talk is based on the following papers:

Kovač, Ž., Platt, T., Sathyendranath, S. (2021).
Sverdrup meets Lambert: Analytical solution for Sverdrup's critical depth.
ICES Journal of Marine Science, 78(4), 1398-1408. doi: 10.1093/icesjms/fsab013.

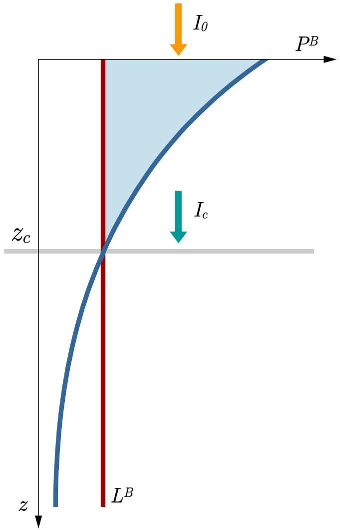
Kovač, Ž., Sathyendranath, S. (2025).
Critical Times for the Critical Depth Theory.
Journal of Geophysical Research, 130 (4), e2024JC021415. doi: 10.1029/2024JC021415.

Kovač, Ž., Sathyendranath, S. (in preparation).
Sverdrup meets Jensen: Unified definition for the compensation depth and the critical depth.

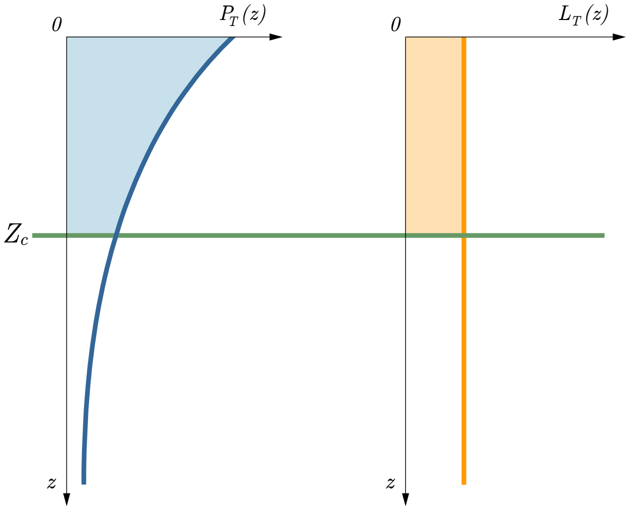
Jensen's inequality (Jensen, 1906)



The compensation depth as defined by Sverdrup (1953)



The classical Critical Depth Criterion (Sverdrup, 1953)



Analytical expressions (Kovač et al., 2021)

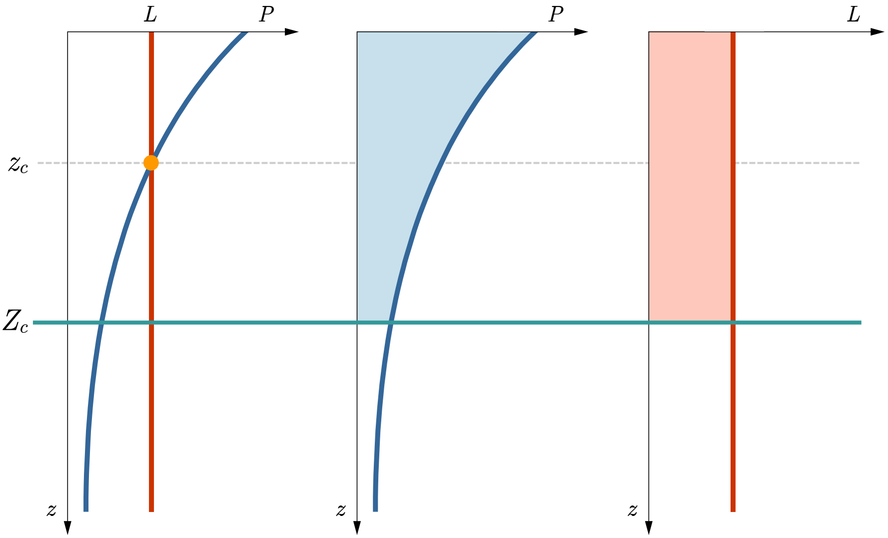
Both the compensation depth and the critical depth have exact solutions:

$$z_c = \frac{1}{K} \ln A$$

$$Z_c = \frac{1}{K} \Big(W_0 \big(- A e^{-A} \big) + A \Big)$$

where the ratio of surface production to losses is given as:

$$A = \frac{\alpha^B I_0}{L^B}$$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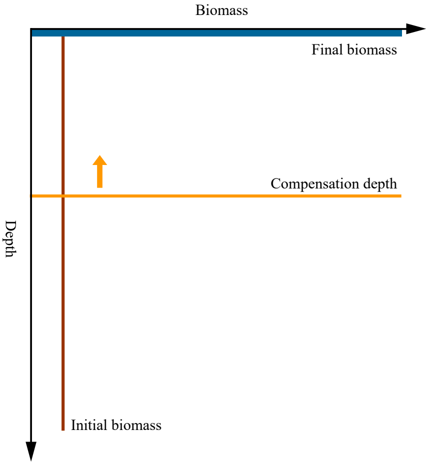
Biooptical feedback (Kovač & Sathyendranath, 2025)



The compensation depth evolves over time according to the following equation:

$$\frac{\partial z_c}{\partial t} = -\frac{k_B}{K_w + k_B B(z_c)} \int_0^{z_c} \frac{\partial B(z')}{\partial t} \, \mathrm{d}z'$$

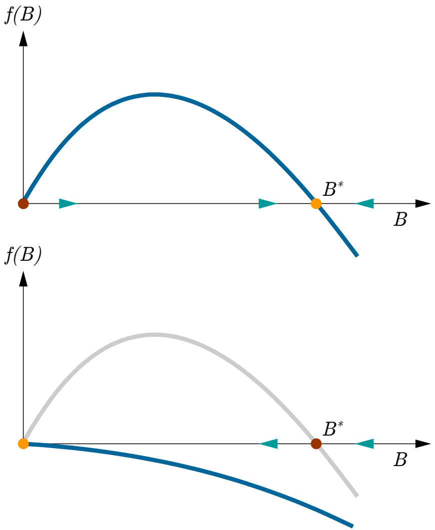
The system has the following constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t} I(z_c(t)) = 0$$

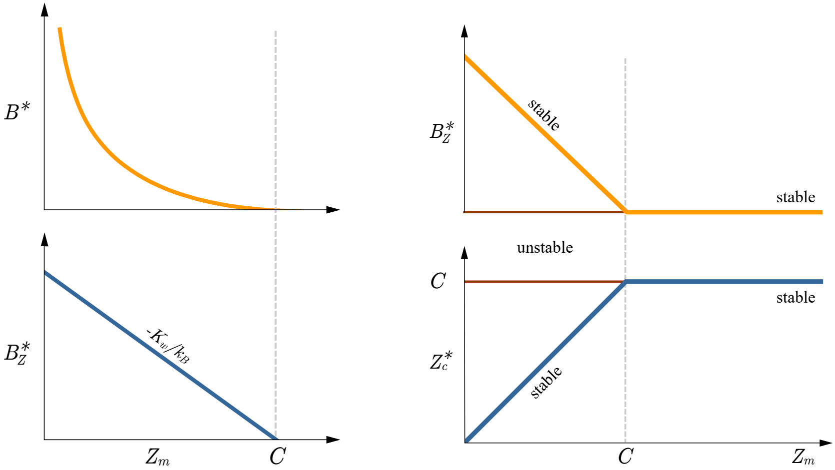
Steady state and the critical depth condition (Kovač & Sathyendranath, 2025)

$$B^* = \frac{K_w}{k_B}\left(\frac{C}{Z_m} - 1\right)$$

$$B_Z^* = \frac{K_w}{k_B}\left(C - Z_m\right)$$



Bio-optical bifurcation (Kovač & Sathyendranath, 2025)



$$C = \frac{1}{K}\Big(W_0\Big(-Ae^{-A}\Big) + A\Big)$$

For active mixing we have the following constant of motion (Kovač et al., 2021):

$$\frac{\mathrm{d}}{\mathrm{d}t}(KZ_c) = 0$$

Switching onto the Lagrangian perspective:

$$\mathrm{d}Z_t = \sigma\,\mathrm{d}W_t$$

W_t is Brownian motion (in the mathematical sense).

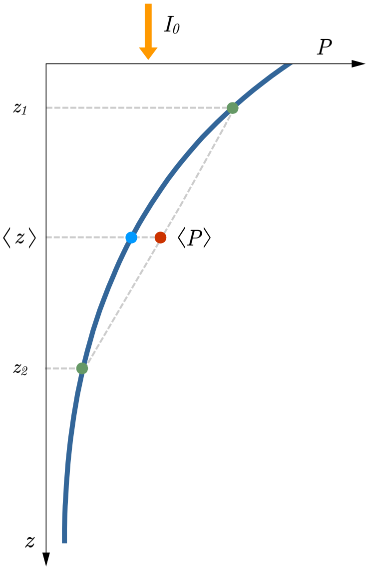
We wish to derive an equation for the production of the cell P_t^B, knowing:

$$P^B(z) = \alpha^B I_0 e^{-Kz}$$

with the depth of the cell given by the solution to the prior equation:

$$Z_t = Z_0 + \sigma W_t$$

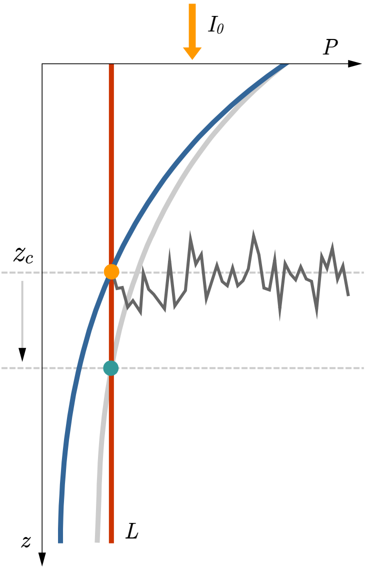
Ito's lemma and Jensen's inequality



$$\mathrm{d}I_t = \frac{\sigma^2 K^2}{2} I_t \,\mathrm{d}t - \sigma K I_t \,\mathrm{d}W_t$$

$$\langle I_t \rangle = I_0 e^{-K Z_0} \exp\left(\frac{\sigma^2 K^2}{2} t\right)$$

New solution for the compensation depth



$$\alpha^B I_0 e^{-K z_c} \exp\left(\frac{\sigma^2 K^2}{2} t\right) = L^B$$

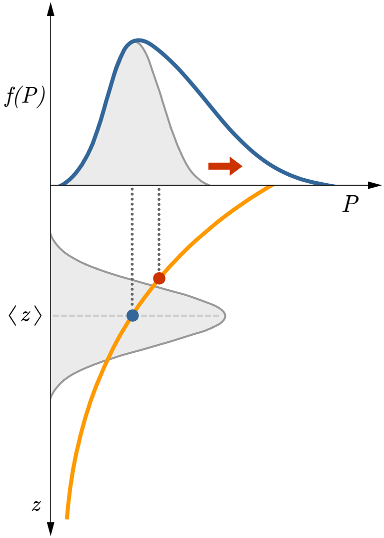
$$z_c = \frac{1}{K} \ln\left(\frac{\alpha^B I_0}{L^B}\right) + \frac{\sigma^2 K}{2} D$$

The above setup results with the following equation for production:

$$\mathrm{d}P_t^B = \frac{\sigma^2 K^2}{2} P_t^B \,\mathrm{d}t - \sigma K P_t^B \,\mathrm{d}W_t$$

The $\frac{\sigma^2 K^2}{2} P_t^B$ term is recognized as Ito drift.

Antifragility recognized in the Ito drift term

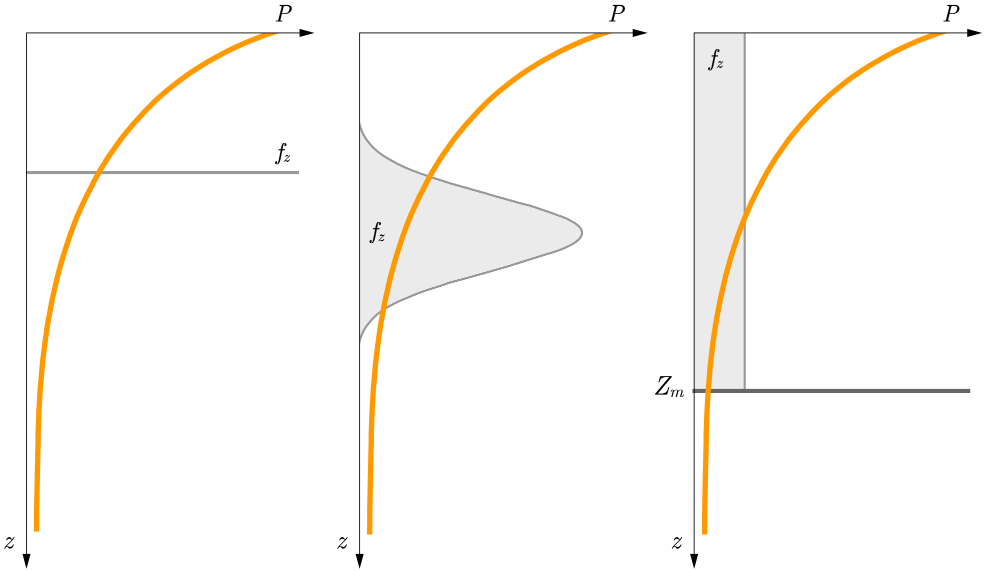


$$\mathrm{d}P_t^B = \frac{\sigma^2 K^2}{2} P_t^B \,\mathrm{d}t - \sigma K P_t^B \,\mathrm{d}W_t$$

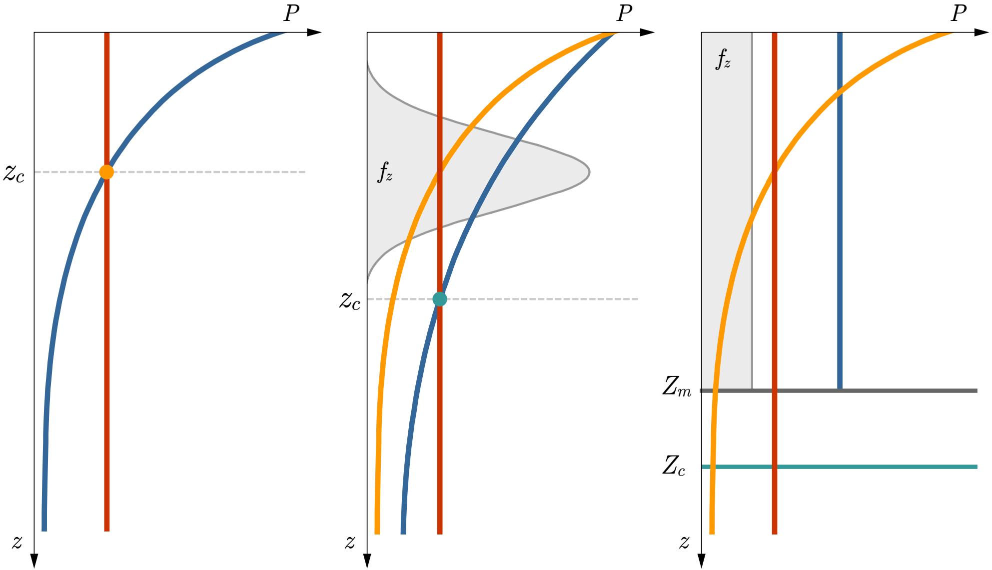
$$f_P(P^B, t) = \frac{1}{P^B \,\sigma K \sqrt{2\pi t}} \exp\left[- \frac{\left(\ln\left(P^B / \alpha^B I_0 e^{-Kz}\right)\right)^2}{2\sigma^2 K^2 t} \right]$$

Unified definition for both the compensation and the critical depth:

$$\int_0^\infty P^B(z') f_z(z', t) \, \mathrm{d}z' = L^B$$



$$P^B(z) < \langle P^B(z) \rangle < \langle P^B \rangle_{Z_m}$$



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Future work

Add sinking to the Z_t equation:

$$\mathrm{d}Z_t = w\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Consider a non-linear photosynthesis irradiance function:

$$P^B(z) = P_m^B\left(1 - \exp\left(\frac{\alpha^B}{P_m^B} I_0 e^{-Kz}\right)\right)$$

The above setup results with the following equation for production:

$$\mathrm{d}P_t^B = \alpha^B K I(Z_t) \exp\left(-\frac{\alpha^B}{P_m^B} I(Z_t)\right) \left[\left(-w + \frac{1}{2}\sigma^2 K \left(1 - \frac{\alpha^B}{P_m^B} I(Z_t)\right)\right) \mathrm{d}t - \sigma \, \mathrm{d}W_t \right]$$

The Ito drift term $-w + \frac{1}{2}\sigma^2 K \left(1 - \frac{\alpha^B}{P_m^B} I(Z_t)\right)$ can now be negative.

Key takeaway: From randomnes to determinism

Due to curvature in the production profile, **randomnes** in the depth of the water parcel **is transferred to a deterministic term** in the production equation.

A new nondimensional number: $\frac{\sigma^2 K}{2} D$.

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Thank you!